Semiclassics

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1 Introduction

Semiclassics is defined by

$$\psi(x,t) = \psi(x_i,t_i)e^{\frac{i}{\hbar}S(\mathbf{x},t;\mathbf{x}_i,t_i)} \tag{1}$$

where S is simply the purely classical principal function from Hamilton-Jacobi theory.

This result can be viewed from two angles. One is starting with quantum mechanics and taking the limit $\hbar \to 0$ which is the usual and an important method. The other is to merely quantize a classically evolving system (I will elaborate on what this means in the article.) [1]

The beautiful result is that these approaches are the same. Semiclassics is this wonderful midpoint where we are just barely short of quantum mechanics while only relying on classical mechanics.

In this short article, I lay out three methods for getting to semiclassics: letting $\hbar \to 0$ in Schrodinger QM, letting $\hbar \to 0$ in Feynmann QM, and quantizing just the ontology of classical mechanics vis-a-vis Dirac. I assume an understanding of Hamilton-Jacobi theory.

2 Schrodinger QM

Start by considering the wavefunction defined by some

$$\psi(x,t) = \psi(x_i,t_i)e^{iK(x,t)/h} \tag{2}$$

Let $\hat{H}$ be a well-ordered Hamiltonian i.e. all $\hat{x}$ are to the left of all $\hat{p}$. All Hamiltonians can be put in this form. This then aligns really well to a classical function $\mathcal{H}(x,p)$. Now note that

$$\left(-i\hbar\frac{\partial}{\partial x}\right)\psi = \frac{\partial K}{\partial x}\psi \tag{3}$$

[1]The latter was done in a paper by Dirac that inspired Feynmann's path integral

$$\left(-i\hbar\frac{\partial}{\partial x}\right)^2\psi = -i\hbar\frac{\partial^2 K}{\partial x^2}\psi + \left(\frac{\partial K}{\partial x}\right)^2\psi \tag{4}$$

As $\hbar \to 0$, the second derivative term reduces to

$$\left(-i\hbar\frac{\partial}{\partial x}\right)^2\psi = \left(\frac{\partial K}{\partial x}\right)^2\psi \tag{5}$$

As we keep applying higher order derivatives, this will keep happening giving us

$$\left(-i\hbar\frac{\partial}{\partial x}\right)^n\psi = \left(\frac{\partial K}{\partial x}\right)^n\psi \tag{6}$$

From this, we can easily generalize to the result

$$f(\hat{x})g(\hat{p})\psi = f(x)g\left(\frac{\partial K}{\partial x}\right)\psi \tag{7}$$

Ultimately then, it is clear that

$$\hat{H}\psi = \mathcal{H}\left(x, \frac{\partial K}{\partial x}, t\right)\psi \tag{8}$$

Finally, the Schrodinger equation will then read

$$-\frac{\partial K}{\partial t} = \mathcal{H}\left(x, \frac{\partial K}{\partial x}, t\right) \tag{9}$$

But this is exactly the Hamilton-Jacobi equation so $K = S(x, t; x_i, t_i)$ taking into account boundary conditions.

3 Feynmann QM

This one is the most well known. Consider the propagator in the path integral formulation.

$$\psi(x_f, t_f) = \psi(x_i, t_i)\int_{x_i, t_i}^{x_f, t_f} \mathcal{D}x\, e^{\frac{i}{\hbar}\mathcal{S}[x]} \tag{10}$$

We simply apply stationary wave approximation to the integrand and get that

$$\psi(x_f, t_f) = \psi(x_i, t_i)e^{iS_c} \tag{11}$$

where

$$S_c = \mathcal{S}[\xi] \tag{12}$$

for $\delta\mathcal{S}|_\xi = 0$. This is exactly the principal function.

4 Generating functions vis-a-vis Dirac

This is, in my opinion, the slickest and most beautiful proof of the semiclassical expression. Generating functions of the first kind $F(q, Q, t)$ generate canonical transformations via the equations

$$p = \frac{\partial F}{\partial q} \tag{13}$$

$$P = -\frac{\partial F}{\partial Q} \tag{14}$$

Consider quantizing these expressions where F is well-ordered in the sense that is a sum of terms of the form $f(q)g(Q)$. It can be shown that

$$\langle q|Q\rangle = e^{\frac{i}{\hbar}F} \tag{15}$$

Consider that

$$\langle q|p|Q\rangle = -i\hbar\frac{\partial}{\partial q}\langle q|Q\rangle \tag{16}$$

$$\langle q|P|Q\rangle = i\hbar\frac{\partial}{\partial Q}\langle q|Q\rangle \tag{17}$$

But given the well-ordered form of F,

$$\langle q|p|Q\rangle = \frac{\partial F}{\partial q}\langle q|Q\rangle \tag{18}$$

$$\langle q|P|Q\rangle = -\frac{\partial F}{\partial Q}\langle q|Q\rangle \tag{19}$$

Combining the expressions above, up to a constant, we get our desired result.

Now our result is a one-liner: the principal function generates time evolution hence our desired semiclassical expression. Namely,

$$\langle q_{t+\tau}|q_t\rangle = e^{\frac{i}{\hbar}S(q_{t+\tau},t+\tau;q_t,t)} \tag{20}$$

Note that $|q_{t+\tau}\rangle \neq e^{-iHt}|q_t\rangle$. It is simply the eigenket for the operator had it evolveed classically. This is the sense in which semiclassics is quantum in form but really classical in spirit.